OSCILLATIONS AND WAVES

- They are produced when a system is perturbed and moved from a stable equilibrium point,
- They result from a balance of elasticity and inertia.-

$$F = -Kx$$

$$-Kx = m\ddot{x}$$

ODE $$\boxed{\ddot{x} = -\frac{k}{m}x}$$ simple harmonic oscillator

$$\ddot{x} = -\omega^2 x \qquad \omega^2 = k/m$$ — elasticity / — inertia

Solution : $x(t) = A\cos(\omega t + \delta)$

A → Amplitude

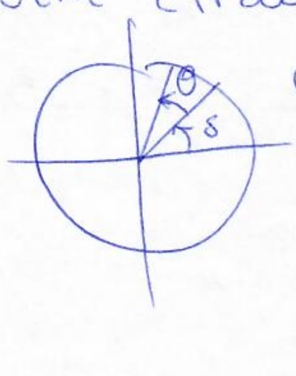
ω → angular frequency

δ → phase (constant)

$x(t)$ is periodic in time $x(t) = x(t+T)$

$T \equiv$ period.

Analogy with circular motion, particle rotating with ω.-



$\theta = \omega t$ -

A & δ are determined from initial conditions

$$\begin{cases} x_0(t_0) = A\cos\delta \\ v_0(t) = -A\omega\sin(\omega t_0 + \delta) \end{cases}$$

Energy of harmonic oscillator

Potential

$$u = \frac{1}{2}Kx^2$$

$$\boxed{u = \frac{1}{2}KA^2\cos^2(\omega t + \delta)}$$

Kinetic

$$T = \frac{1}{2}mv^2 = \frac{1}{2}m\dot{x}^2$$

$$\dot{x} = -A\omega\sin(\omega t + \delta)$$

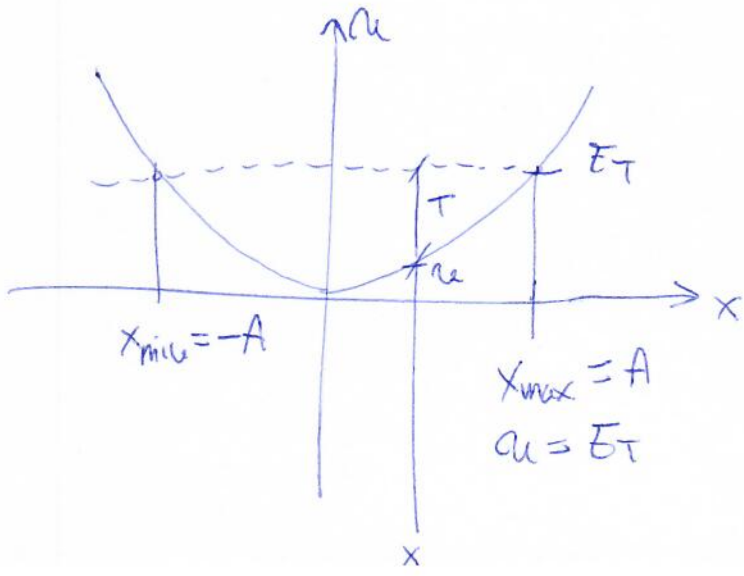
$$\boxed{T = \frac{1}{2}mA^2\omega^2\sin^2(\omega t + \delta)}$$

$$E_{TOTAL} = T + u$$

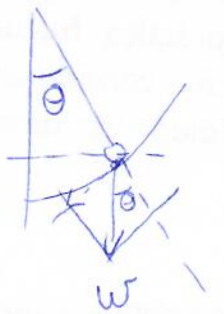
$$E_T = \frac{1}{2}KA^2\cos^2(\omega t + \delta) + \frac{1}{2}mA^2\omega^2\sin^2(\omega t + \delta)$$

$$E_T = \frac{1}{2}KA^2\cos^2(\;) + \frac{1}{2}\cancel{m}A^2\frac{K}{\cancel{m}}\sin^2(\;)$$

$$\boxed{E_T = \frac{1}{2}KA^2}$$



The pendulum



$$-mg\sin\theta = m\frac{d^2s}{dt^2} \qquad s = \text{arc}$$

$$\theta = \frac{s}{l}, \quad s = \theta \cdot l$$

$$\ddot{s} = \ddot{\theta}\, l$$

$$-g\sin\theta = \frac{d^2s}{dt^2}$$

$$-g\sin\theta = l\frac{d^2\theta}{dt^2} = l\ddot{\theta}$$

$$\boxed{-\frac{g}{l}\sin\theta = \ddot{\theta}}$$ Pendulum equation

If small oscillations ↓

$$\sin\theta \simeq \theta \quad \text{when } \theta \ll$$

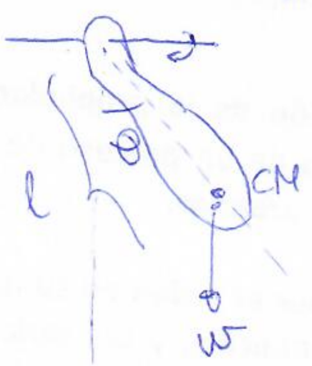
$$\boxed{-\frac{g}{l}\theta = \ddot{\theta}}$$ The harmonic oscillator

$$\omega^2 = \frac{g}{l}$$

$$T = \frac{2\pi}{\omega} = 2\pi\sqrt{l/g}$$ – can be used for g measurements

– it does not depend on amplitude.

The physical pendulum



$\tau = r \cdot F = I \cdot \alpha$

τ because weight -

$-Mgl \sin\theta = I\ddot{\theta}$

$$\boxed{\ddot{\theta} = -\frac{Mgl}{I}\sin\theta}$$ Physical Pendulum

If small oscillations

$\theta <<$ $\sin\theta \simeq \theta$

$$\boxed{\ddot{\theta} = -\frac{Mgl}{I}\theta}$$ harmonic oscillations

$T = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{I}{mgl}}$ period

Damped oscillations

$F_{damp} = -bv$ ← typical, standard assumption

$-kx - b\dot{x} = m\ddot{x}$

$$\boxed{m\ddot{x} + kx + b\dot{x} = 0}$$ ODE, to be solved -

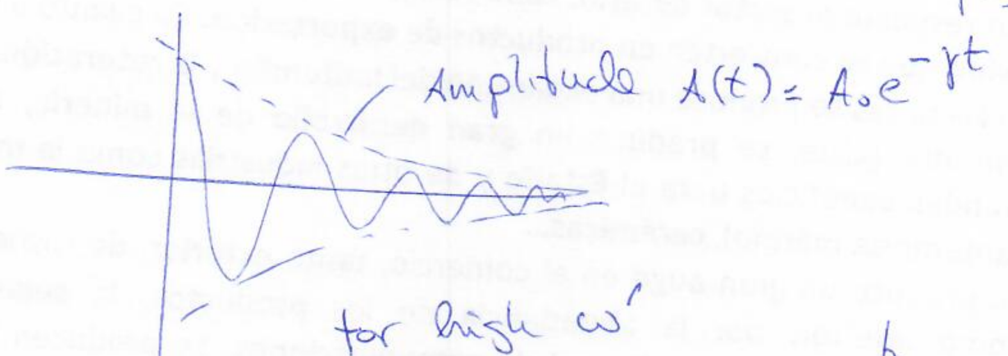
solution when $b <<$ → under-damped oscillator

$x = A_0 e^{-b/2m t} \cdot \cos(\omega' t + \delta)$

$\gamma = \frac{b}{...}$ ≡ damping coefficient

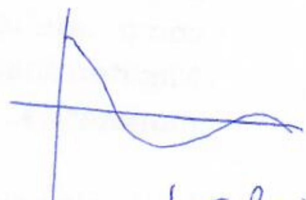
$\tau \equiv$ 'extinction time' $\equiv$ 'time constant'
time needed for reducing amplitudes a factor $1/e$.

$$\omega' = \sqrt{\omega_0^2 - \left(\frac{b}{2m}\right)^2} \quad \text{with} \quad \omega_0 = \sqrt{\frac{k}{m}} \equiv \text{natural or undamped frequency.}$$



Amplitude $A(t) = A_0 e^{-\gamma t}$

for high ω'

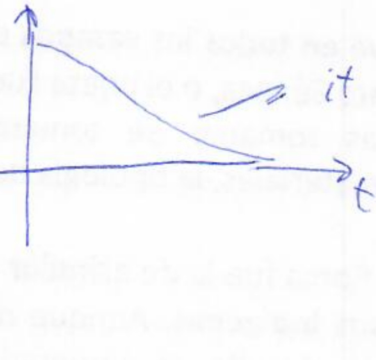


for low ω'

b) when b (γ) is large.

ω' is even lower.

when $b = 2m\omega_0 \rightarrow \omega' = 0 \equiv$ critical damping



it never will cross the axis.

$\boxed{\text{Forced oscillations}}$ (or 'driven' oscillators)

The damped oscillator does not move freely, but subject to an external force $F(t)$

Typically, $F(t) = F_0 \cos(\omega t)$

Hence,

$$\boxed{m\ddot{x} + b\dot{x} + \underbrace{K}_{m\omega_0^2} x = F_0 \cos(\omega t)} \quad \text{ODE}$$

driving freq.

One can solve this ODE. There is a transient part that depends on initial condition, similar to the damped oscillator. But later on, it reaches a steady solution, the same for every initial condition,

$$x(t) = A \cos(\omega' t - \delta)$$

driving frequency; $\omega' = \omega$ in steady solution

with

$$\begin{cases} A = \dfrac{F_0}{\sqrt{m^2(\omega_0^2 - \omega^2)^2 + b^2\omega^2}} \\[2ex] tg\,\delta = \dfrac{b\omega}{m(\omega_0^2 - \omega^2)} \end{cases}$$

$\omega_0 = \frac{K}{m}$ natural frequency.

Notably, when $\underset{\text{driving}}{\omega} = \omega_0 \Rightarrow \omega_{resonant}$, it is called resonant case.-

$$\begin{cases} A \text{ is maximum} \\ \delta = \pi/2 \end{cases}$$

$$\begin{cases} \text{if } \omega \ll \omega_0;\ \delta \sim 0 \\ \text{if } \omega \gg \omega_0;\ \delta \sim \pi \end{cases}$$

In this resonant case: -

$$v = \frac{dx}{dt} = \dot{x} = -A\omega \sin(\omega t - \delta)$$

$$\text{if } \delta = \pi/2;\quad v_r = -A\omega \sin(\omega t - \pi/2)$$

$$v_r = +A\omega \cos(\omega t)$$

$v_{resonance}$ is in phase with driving force → the particle moves in the same direction than the force → maximum energy transport

$\boxed{\text{Averaged power}}$

$P(t)$ = instantaneous force

$$P(t) = F \cdot v = -A\omega F_0 \cos(\omega t) \sin(\omega t - \delta)$$

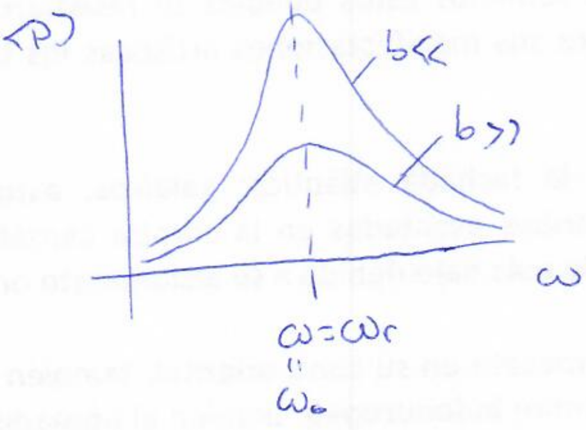
using $\sin(\theta_1 - \theta_2) = \sin\theta_1 \cos\theta_2 - \cos\theta_1 \sin\theta_2$

$$P(t) = A\omega F_0 \sin\delta \cos^2\omega t - A\omega F_0 \cos\delta \cos(\omega t)\sin(\omega t)$$

Averaging $\langle P \rangle = \dfrac{\int_0^T P(t)\,dt}{T} = \dfrac{1}{2} A\omega F_0 \sin\delta$

Replacing $\sin\delta = \dfrac{b\omega}{\sqrt{m^2(\omega_0^2-\omega^2)^2 + b^2\omega^2}}$

$$\langle P \rangle = \frac{1}{2}\left[\frac{b\omega^2 F_0^2}{\sqrt{m^2(\omega_0^2-\omega^2)^2 + b^2\omega^2}}\right]$$



The averaged power grows as $\omega \to \omega_0$

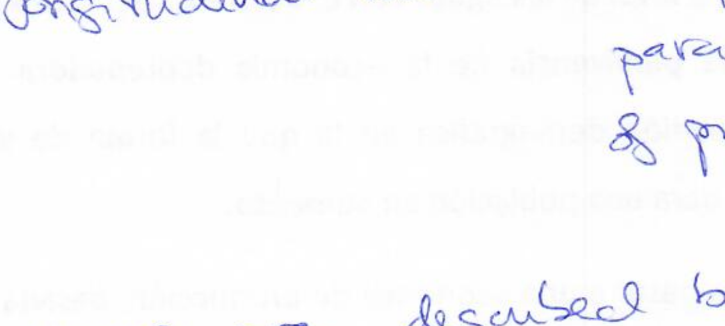
WAVES

- A travelling perturbation that is periodic both in time and space.-
- It transports energy and linear momentum without mass.-

{ Mechanical waves: they travel through a medium
{ Electromagnetic waves: they are oscillating fields and they do not need any material medium

{ Transverse waves: the perturbation is perpendicular to the direction of propagation
{ Longitudinal waves: the perturbation travels parallel to the direction of propagation.

Waves are described by wave-functions.-



$\Delta x = v\,\Delta t$ $\quad$ v = speed of the wave

$f(x,t) = f(x+\Delta x, t+\Delta t)$

$f(x,t) = f(x - vt)$ (I)

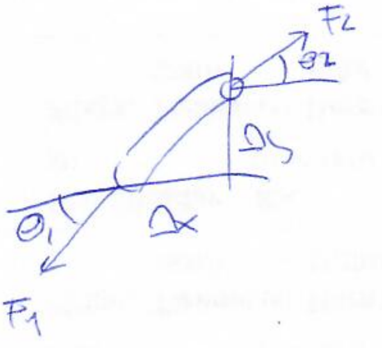
$f(x,t) = f(x + vt)$ (II)

moving to the left. (II)

moving to the right (I)

Equation of a wave ≡ wave function

in a particular case: an oscillating rope.



~~De~~ · This segment of the rope will oscillate vertically.-

$$\sum F = F\sin\theta_2 - F\sin\theta_1$$

$$\sim F(tg\theta_2 - tg\theta_1)$$

using $tg\,\theta \sim \sin\theta$

$$\sum F \cong F\left(\frac{\partial y_2}{\partial x_2} - \frac{\partial y_1}{\partial x_1}\right) = \text{'}ma\text{'}$$

'a'

$$F \cdot \Delta_{slope} = \text{"mass"} \cdot \left(\frac{\partial^2 y}{\partial t^2}\right)$$

lineal density $\rightarrow$ $\mu = \frac{m}{L} \Rightarrow mass = m = \mu \cdot L = \mu\,\Delta x$

$$F\,\Delta_{slope} = \mu\,\Delta x \cdot \frac{\partial^2 y}{\partial t^2}$$

$$F\,\frac{\Delta slope}{\Delta x} = \mu\,\frac{\partial^2 y}{\partial t^2}$$

$$F\,\frac{\partial}{\partial x}\underbrace{\left(\frac{\partial y}{\partial x}\right)}_{slope} = \mu\,\frac{\partial^2 y}{\partial t^2}$$

$$\boxed{\frac{\partial^2 y}{\partial x^2} = \frac{\mu}{F}\,\frac{\partial^2 y}{\partial t^2}}$$

ODE wave equation

solution: a wave travelling $v^2 = \frac{F}{\mu}$

The general wave equation is:

$$\boxed{\frac{\partial^2 y}{\partial x^2} = \frac{1}{v^2}\frac{\partial^2 y}{\partial t^2}}$$ wave equation :-

The solutions are of the form $f(x-vt)$

Mechanical waves such as 'ropes' $\rightarrow$ $v = \sqrt{F/\mu}$

Sonic waves | in a fluid : $v = \sqrt{B/\rho}$ $B \equiv$ compressibility, $\rho \equiv$ density

| in a gas for p & T $v = \sqrt{\frac{\gamma R T}{M}}$

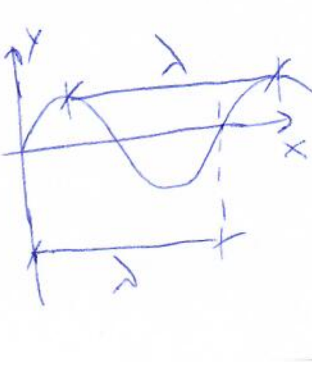
Electromagnetic waves $v = \sqrt{\frac{1}{\mu_0 \epsilon_0}}$

The solutions can be diverse.
The simples solution (not the only one) is
the so-called 'harmonic wave'

$$y(x,t) = A \sin(Kx - \omega t) = A \sin[(K)(x-vt)]$$

$A \equiv$ amplitude
$\omega \equiv K v$; $K = \frac{2\pi}{\lambda}$ 'wave number'

$$v = \frac{\lambda}{T} = \lambda \nu$$



$\lambda \equiv$ wave lenght $T \equiv$ period.

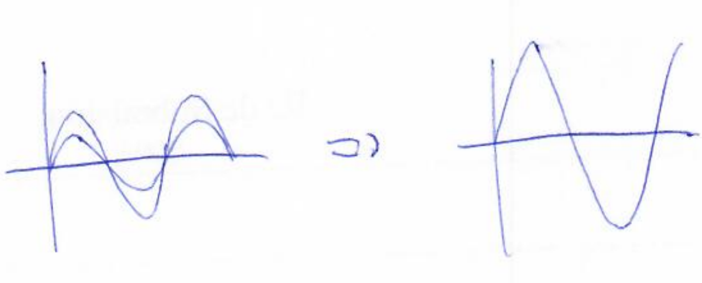
Principle of superposition

- A consequence of the linearity of the wave function:

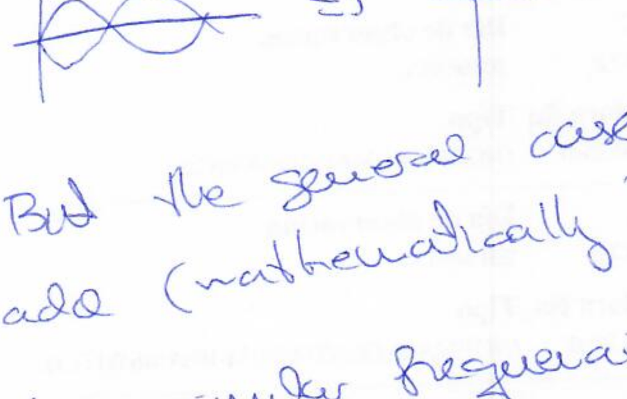
Given two solutions, $y_1(x,t)$ and $y_2(x,t)$

$$y_3(x,t) = \underbrace{\alpha y_1 + \beta y_2}_{\text{linear combination}}$$ is also solution

Adding waves is called 'interference'



constructive when in phase



destructive when in phase

But the general case → we need to add (mathematically) the two functions.

When similar frequencies and same amplitude

$$\psi(x,t) = A(x,t)\sin(kx-\omega t)$$

$$\psi = \underbrace{2\cos(k'x-\omega' t)}_{\text{the amplitude, moves with 'group velocity'}}\underbrace{\sin(kx-\omega t)}_{\text{the wave travels with 'phase velocity'}}$$

The group velocity is the velocity of the 'amplitude', of the 'envelope'.
It carries the information = signal velocity.

The phase velocity $v_p = \frac{\omega}{k}$ it is just the propagation of the crests, without information

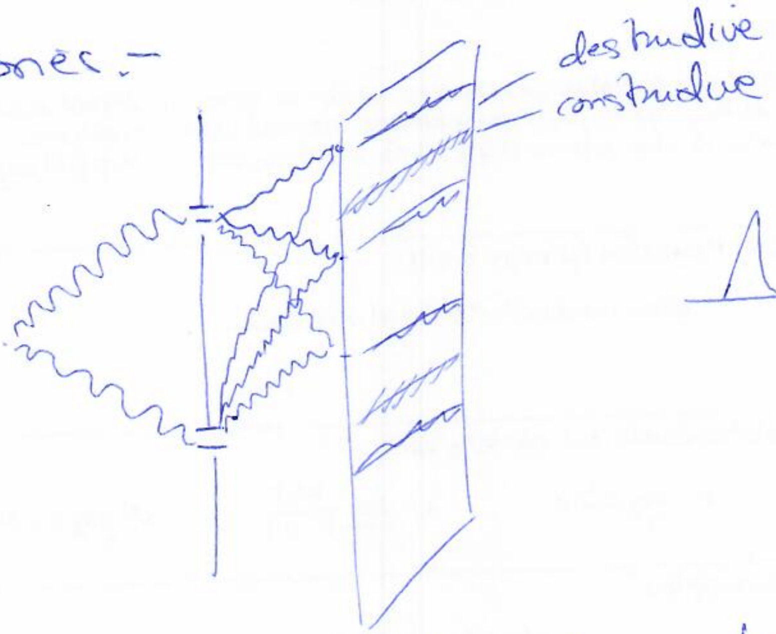
When the wave reaches a different medium than vacuum $v = \frac{c}{n}$; $n = \frac{c}{v}$

Vacuum $n = 1$ $v = c$ air $n = 1{,}0003$
Typically $n > 1$; v_{phase} decreases water $n \simeq 1{,}33$
But it may happen $v_{phase} > c$! $n < 1$.

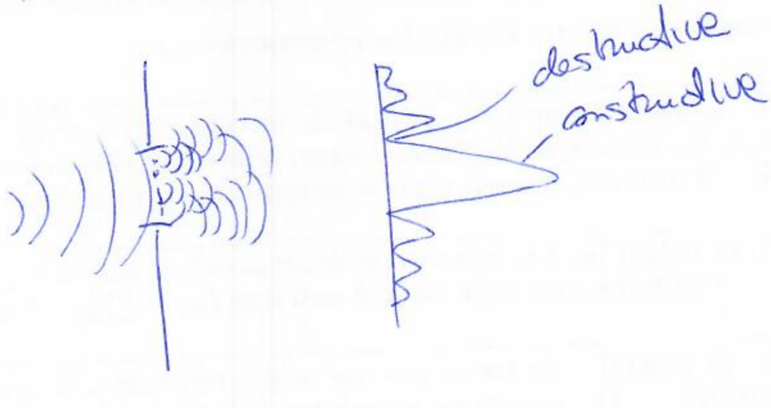
water $v_{light} = 0{,}75\,c$
Some e^-'s can travel faster than that with enough energy $\to$ Cerenkov radiation.

Interference

Two waves originated in phase, can arrive at a given point with different phases because they travelled along different trajectories:-



when waves pass through a hole of size similar to λ → 'diffraction'



Standing waves

- A wave travelling can find an obstacle, a wall and can be reflected.-

bouncing back.-

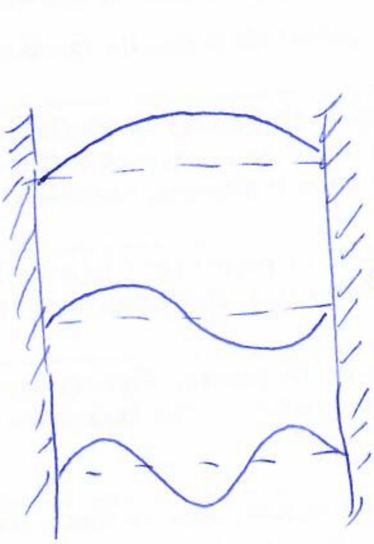
- If the wave gets confined between two fixed walls one can get a stationary oscillating schema ≡ 'standing waves'

Because we get a final schema resulting from the superposition principle, when waves travelling from left to right, and vice-versa interfere.

The resulting schema:

$$L = n \frac{\lambda_n}{2}$$

$n = 1, 2, 3, \ldots$



$n=1 \quad L = \lambda/2$

$n=2 \quad L = \lambda$

$n=3 \quad L = \frac{3}{2}\lambda$

The final schema is the sum of all above

The resonance condition is reached when

$$\frac{2L}{v} = nT \Rightarrow \boxed{\nu = n\frac{v}{2L}}$$ natural freq's or resonant freq's

The final motion : a linear combination of the natural freq's.-

* A standing wave will have equation:

$$y_n(x,t) = A_n \sin(k_n x)\cos(\omega_n t + \delta_n)$$

$$k_n \equiv \frac{2\pi}{\lambda_n}$$

$$A_n(x) = A_n \sin(k_n x)$$

Every point at position 'x' ~~moves~~ oscillates harmonically but it does not travel.-

* The final motion : superposition of several standing waves.-

$$Y_T = \sum y_n(x,t) = \sum A_n \sin(k_n x)\cos(\omega_n t + \delta_n)$$

↳ i.e. like Fourier expansion.-